Journal of Structural Geology, Vol. 15, Nos 3–5, pp. 413 to 422, 1993
Printed in Great Britain

0191–8141/93 $06.00+0.00

The deformation matrix for simultaneous simple shearing, pure shearing and volume change, and its application to transpression–transtension tectonics

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(*Received* 23 *December* 1991; *accepted in revised form* 22 *July* 1992)

Abstract—Simultaneous simple shearing and pure shearing, with or without additional volume change, can be combined into a single, upper triangular deformation matrix. The off-diagonal term, Γ, is named the effective shear strain, and is a simple function of the pure shearing and simple shearing components. A three-dimensional deformation matrix for the simultaneous combination of coaxial deformation, with or without additional volume change, and up to three simple shearing systems with mutually orthogonal shear planes is also presented. By using this matrix, one can easily extract the various properties of incremental as well as finite strain, and the progressive as well as finite rotation of passive markers during deformation.

The case of transpression–transtension is revised, using the unified deformation matrix. The orientation of the major axis of the strain ellipsoid (λ_1) is always horizontal if the deformation is transtensional, switches from horizontal to vertical during transpressional wrenching ($1 > W_k > 0.81$ for constant vorticity deformations), and is always vertical for highly transpressional deformations ($W_k \leq 0.81$). For transpression, material lines initially rotate towards the horizontal shearing direction, but generally turn to rotate towards the vertical axis after a certain strain. For transtension, all material lines rotate towards a direction in the horizontal plane which is oblique to the shearing direction.

INTRODUCTION

IT IS simple and useful to think of rock deformation in terms of simple shear, pure shear and volume change (dilation), and these deformations have frequently been applied to explain deformation structures observed in rocks. Even under fairly simple conditions, however, plane strain deformation is likely to be a combination of two or more of these components. For instance, combinations of a simple shear and volume change are probably very common, even in 'perfect' shear zones of constant thickness and with undeformed or homogeneously deformed wallrocks (Ramsay & Graham 1970). More general shear zones do not obey these constraints, and have additional components of pure shear. Hence, simple shear, volume change and pure shear are only end-members of a wide range of deformation types, and each alone can rarely explain the deformation structures observed in rocks. The acknowledgment of this fact led to a series of fundamental articles in the 1980s which treated finite deformation as a combination of simple shear, pure shear and/or volume change (Ramsay 1980, Coward & Kim 1981, Kligfield *et al.* 1981, Sanderson 1982, Coward & Potts 1983, Sanderson & Marchini 1984). A mathematically convenient order of superposition of simple shear, pure shear and/or volume change was assumed in these works to model finite strain. The order of superposition was chosen without reference to the actual geological deformation history, which was not considered in most of these articles.

It is reasonable to assume, and in some cases it can be demonstrated (e.g. Passchier & Urai 1988, Wallis 1992), that natural, progressive deformation generally occurs by *simultaneous* pure shearing, simple shearing, and/or dilating (progressive volume change). (The *-ing* suffix will be used where it is essential to emphasize the kinematic aspect of deformation, as suggested by Means 1990, and not merely the finite state of deformation.) Hence, if deformation history is a concern, an approach different from that of discrete strain factorization is required. The theory needed is available in a continuum mechanics framework (e.g. Malvern 1969), and is elegantly laid out by Ramberg (1975) in terms of pure shearing and simple shearing strain rates. However, since many geologists are not very familiar with the theory of continuum mechanics, we present a convenient way of combining these three deformation end-members into a single, unified deformation matrix in terms of simple shearing and pure shearing–dilating components. We also extend the discussion to three dimensions in which pure shearing becomes three-dimensional coaxial deformation, and in which several orthogonal sets of simultaneous simple shearing deformations may occur. The application of this theory to structural modelling is finally demonstrated for transpressional–transtensional deformation.

THE DEFORMATION MATRIX

For homogeneous deformation, the matrix **D** describes a linear transformation relating the undeformed vector or point (**x**) in a Cartesian co-ordinate system to its position after deformation (**x**′):

$$\mathbf{x}' = \mathbf{D}\mathbf{x} \tag{1}$$

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(e.g. Flinn 1979). For plane strain this matrix transformation is equal to the transformation equations

$$x_1' = D_{11}x_1 + D_{12}x_2$$
$$x_2' = D_{21}x_1 + D_{22}x_2,$$

where D_{ij} are the components of **D**, and where any translation involved is neglected. Knowing the deformation (gradient) matrix **D**, the orientation and geometry of the strain ellipse are easily calculated (see the Appendix). The deformation matrices for simple shear and pure shear with or without dilation are:

$$\mathbf{D}_{ss} = \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{D}_{ps,\Delta} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}, \tag{2}$$

where γ is the shear strain, and k_1 and k_2 are the extension–contraction along the x_1 and x_2 axis, respectively. By definition (e.g. Ramsay & Huber 1983), $k_1 = 1/k_2$ for pure shear, hence if $k_1k_2 \neq 1$, a volume change is involved in addition to the pure shear in the matrix $\mathbf{D}_{ps,\Delta}$. A familiar, anisotropic volume change compatible with both simple shear and pure shear is (e.g. Sanderson 1976, Ramsay 1980):

$$\mathbf{D}_{\Delta} = \begin{bmatrix} 1 & 0 \\ 0 & 1+\Delta \end{bmatrix}, \tag{3}$$

where Δ is the volume change. Because the matrices D_{ps} and D_{Δ} are both diagonal, the pure shear and volume change can easily be combined into one matrix by simple matrix multiplication:

$$\mathbf{D}_{ps,\Delta} = \mathbf{D}_{ps}\mathbf{D}_{\Delta} = \mathbf{D}_{\Delta}\mathbf{D}_{ps}$$
$$= \begin{bmatrix} k & 0 \\ 0 & (1+\Delta)k^{-1} \end{bmatrix} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}, \tag{4}$$

where $1 + \Delta = \det \mathbf{D}_{ps,\Delta} = k_1k_2$. In general, however, matrix multiplication is non-commutative, and the simultaneous combination of simple shearing and pure shearing–dilating into a single, unified matrix is not a straightforward problem:

$$\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}\begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & \gamma \\ 0 & 1 \end{bmatrix}\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}. \tag{5}$$

The left-hand side of (5) is, mathematically, a simple shear deformation followed by pure shear and/or volume change, and the right-hand side is a pure shear and/or volume change followed by simple shear. Both the deformation path and the finite strain are different for the two different orders of deformations, and different from simultaneously acting pure shearing and simple shearing.

In geology, the *finite* strain is commonly known, and therefore fixed. Any finite deformation, where only the initial and final positions of points or vectors (**x**) are considered, can be factorized in an infinite number of ways into two or more deformations, each represented by a single deformation matrix. A common factorization is decomposition into rotation and stretch (Elliott 1972), another is factorization into simple shear, pure shear

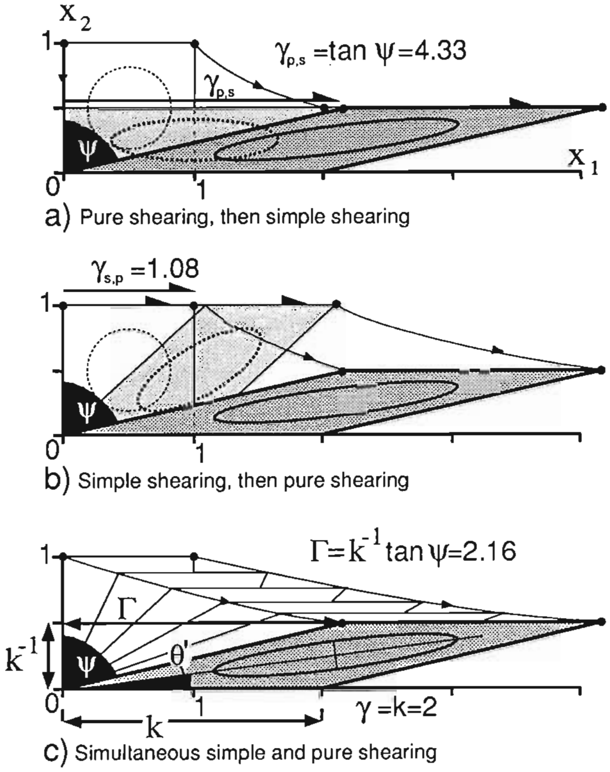

Fig. 1. Three ways to arrive at the same finite state of deformation. (a) A pure shear is followed by a simple shear. (b) A simple shear is followed by a pure shear, and (c) a simultaneous pure shearing and simple shearing. Note that the respective shear strains involved have been chosen so that the same finite strain is achieved in all cases, whereas the pure shear component is the same (k = 2) in all cases.

and/or dilation, as discussed in this article (Fig. 1). Although any such factorization is, mathematically, a sequence of linear transformations (deformations), they have been used merely as a way of representing the finite strain in a shear zone by two different parameters. However, the shear strain on each side of (5) must be different for the finite strain to be the same:

$$\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}\begin{bmatrix} 1 & \gamma_{s,p} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \gamma_{p,s} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} = \mathbf{D}, \tag{6}$$

where $\gamma_{p,s} \neq \gamma_{s,p}$ for non-zero shear strain values.

For (6) to hold true, the relationship between the two shear strains must be

$$\gamma_{p,s} = (\gamma_{s,p})(k_1/k_2). \tag{7}$$

It may be noted that $\gamma_{p,s} = \tan(\psi)$ (see Fig. 1), and the factorization on the left-hand side of (6) is therefore commonly preferred (e.g. Sanderson 1982).

Any factorization of the deformation matrix can be used for treating finite strain (shape and orientation of the strain ellipsoid) and finite rotation and change of length of material lines (passive markers). However, if one is interested in the strain history, a strain factorization has significant implications. For example, an initially arbitrarily oriented line will rotate to the exact same position and experience the same finite extension irrespective of the strain factorization chosen, but the strain history of the line (i.e. shortened, then extended, etc.) will be different. In fact, together with some simplifying assumptions, this can be utilized to extract the vorticity (see below) from deformed rocks, as outlined by Passchier (1990). Since an arbitrary factorization of deformation prevents a realistic modelling of the deformation history, and since progressive deformation generally involves simultaneous simple shearing and pure shearing, a deformation matrix which combines simultaneous pure shearing and simple shearing is often needed. Such a deformation matrix was derived by Ramberg (1975, equation 38) in a continuum mechanics framework:

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{bmatrix} \exp(\dot{\varepsilon}_{x_1} t) & \dot{\gamma}\dfrac{\exp(\dot{\varepsilon}_{x_1} t) - \exp(-\dot{\varepsilon}_{x_1} t)}{2\dot{\varepsilon}_{x_1}} \\ 0 & \exp(-\dot{\varepsilon}_{x_1} t) \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad (8)$$

where $\dot{\varepsilon}_{x_1}$ is the rate of pure shearing, i.e. the extension rate parallel to the shear direction, $\dot{\gamma}$ is the rate of simple shearing and t is time. Using continuum mechanics nomenclature (Malvern 1969, Means 1990), $\dot{\varepsilon}_{x_1} = L_{11}$ and $\dot{\gamma} = L_{12}$, where L_{ij} are the velocity gradients.

Although matrix (8) may be used in its present form (e.g. Ramberg & Ghosh 1977, Ingles 1983, Kligfield *et al.* 1985), a somewhat simpler, time-independent version of (8) may be derived as indicated by Coward & Kim (1981) and Merle (1986):

$$\mathbf{D} = \begin{bmatrix} k & \Gamma \\ 0 & k^{-1} \end{bmatrix} = \begin{bmatrix} k & \dfrac{\gamma(k - k^{-1})}{2 \ln k} \\ 0 & k^{-1} \end{bmatrix}, \quad (9)$$

where the off-diagonal (rotational) term is a function of the pure shearing and simple shearing components, and may be termed Γ (effective shear strain). Here, $\gamma = L_{12}t$, and $k = L_{11}t$.

There is a unique orientation and geometry of the finite strain ellipse for a certain simultaneous combination of simple shearing and pure shearing, as shown in Fig. 2. The result of using (9) can be approximated by performing a large number of successive pure shear and simple shear increments, as shown by Ramberg (1975). The numerical value of γ in (9) is always between those of $\gamma_{p,s}$ and $\gamma_{s,p}$, and their relationship is:

$$\gamma_{p,s} = k\Gamma = k^2(\gamma_{s,p}). \quad (10)$$

The effect of various simultaneous combinations of simple shearing and pure shearing on displacement and rotation in the shear direction can be extracted from (9), as illustrated in Fig. 3, which also shows the relationship between the shear strain ($\gamma_{p,s}$) used by Sanderson (1982) and the shear strain (γ) for simultaneous simple shearing and pure shearing. Hence, if the geometry and orientation of the strain ellipse (R and θ') can be estimated from a shear zone, then γ, $\gamma_{p,s}$, $\gamma_{s,p}$, Γ and k can be found by using Fig. 2 and/or equation (10). If the angular shear strain in the x_1 direction (ψ) can be measured, then either γ or $\gamma_{p,s}$ can be found from the relationships

$$\tan \psi = k\Gamma = \gamma_{p,s} \quad (11)$$

or

$$\gamma = \frac{2 \ln k}{k^2 - 1}\gamma_{p,s} \quad (12)$$

(see Fig. 1). Hence, since k is independent of the deformation history, finding γ from $\gamma_{p,s}$ or vice versa is trivial.

To account for dilation, the unified deformation matrix, $\mathbf{D}$, becomes

$$\mathbf{D} = \begin{bmatrix} k_1 & \Gamma \\ 0 & k_2 \end{bmatrix} = \begin{bmatrix} k_1 & \dfrac{\gamma(k_1 - k_2)}{\ln(k_1/k_2)} \\ 0 & k_2 \end{bmatrix}. \quad (13)$$

For simultaneous simple shearing and dilating perpendicular to the shear zone, the relationship between the orientation of the strain ellipse and the shear plane (shear zone boundary) is given by Fig. 4. This situation

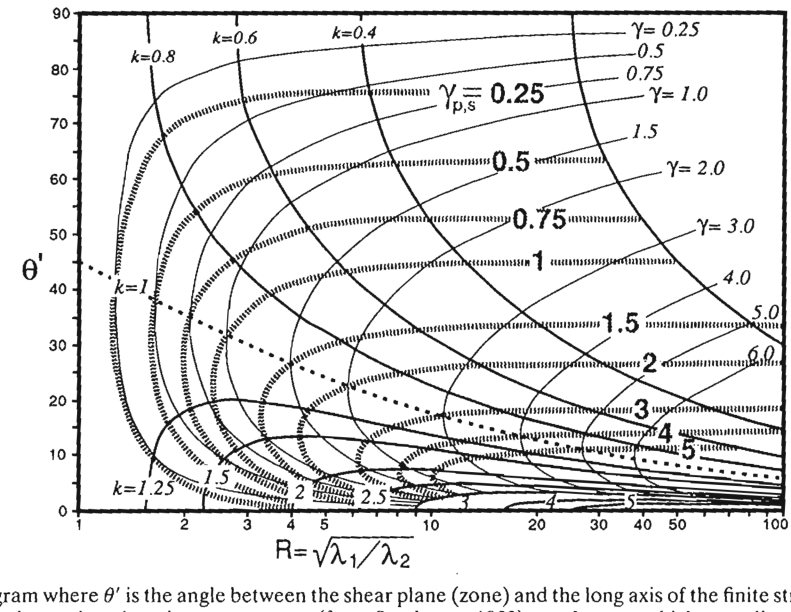

Fig. 2. R–θ' diagram where θ' is the angle between the shear plane (zone) and the long axis of the finite strain ellipse, and R is the strain axis ratio. $\gamma_{p,s}$ contours (from Sanderson 1982) are shown as thick, grey lines.

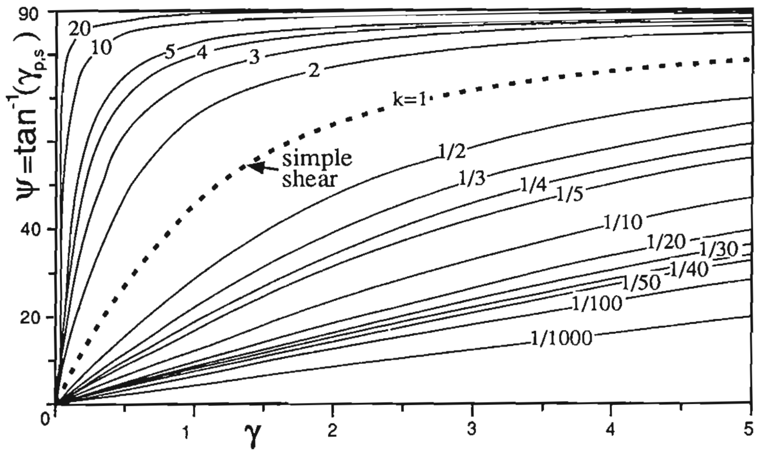

Fig. 3. Variation in angular shear strain (ψ) for simultaneous simple shearing and pure shearing. The displacement across such a pure shear–simple shear zone = finite vertical distance · tan ψ.

was treated in detail by Ramsay (1980, equations 17–24) for finite strain, who based his discussion on factorization of the finite deformation as simple shear followed by volume change (14):

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1+\Delta \end{bmatrix} \begin{bmatrix} 1 & 0 & \gamma_{s,\Delta} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \gamma_{s,\Delta} \\ 0 & 1 & 0 \\ 0 & 0 & 1+\Delta \end{bmatrix}. \quad (14)$$

If one is interested in the deformation history, however, the deformation matrix for simultaneous simple shearing and progressive volume change, which can be derived from (9) to be

$$\begin{bmatrix} 1 & 0 & \dfrac{\gamma\Delta}{\ln(1+\Delta)} \\ 0 & 1 & 0 \\ 0 & 0 & 1+\Delta \end{bmatrix}, \quad (15)$$

is more useful. As above, there is a difference between $\gamma_{s,\Delta}$ in (14) and the γ in (15). Substituting the relationship

$$\gamma_{s,\Delta} = \frac{\Delta}{\ln(1+\Delta)}\gamma \quad (16)$$

into Ramsay's formulas gives the corresponding formulas for simultaneous simple shearing and progressive volume change.

THE THREE-DIMENSIONAL DEFORMATION MATRIX

The discussion above can be extended to three dimensions, and we will discuss the simultaneous combination

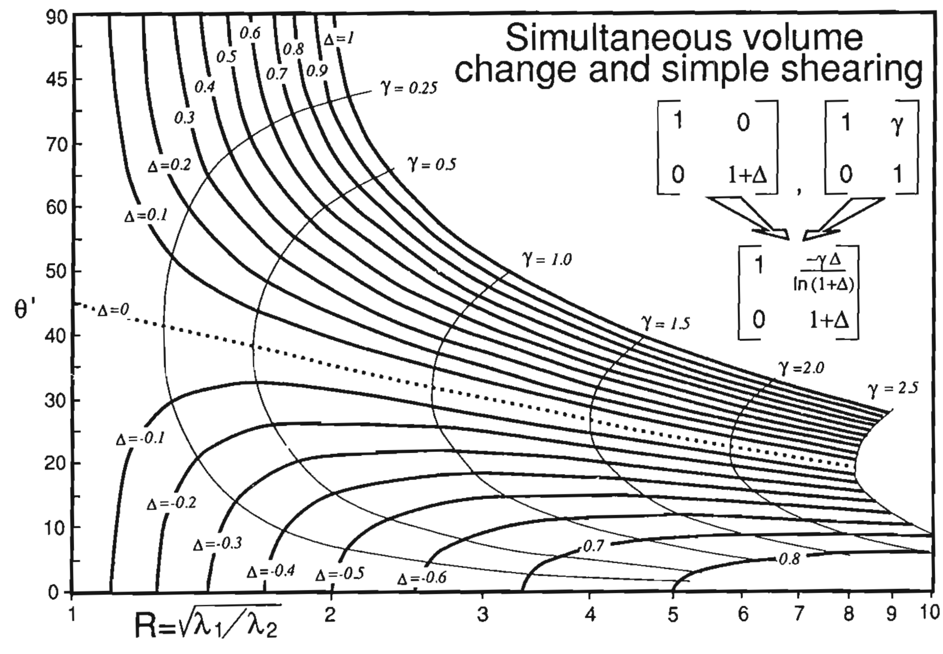

Fig. 4. Similar to Fig. 2 but for simultaneous simple shear and volume change. Note that this diagram is different from a similar diagram shown by Kligfield *et al.* (1981, fig. 12) and Ramsay & Huber (1983, p. 50) merely because of the different meaning of γ in these works.

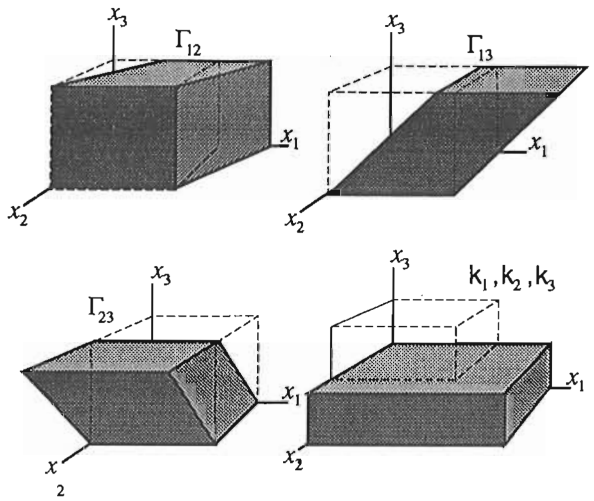

Fig. 5. The deformations accounted for in the three-dimensional deformation matrix (equation 17).

of three orthogonal simple shear systems and a three-dimensional pure shear with or without additional volume change. This general deformation covers a number of realistic deformations in the crust, and can be expressed by the matrix

$$\mathbf{D} = \begin{bmatrix} k_1 & \Gamma_{12} & \Gamma_{13} \\ 0 & k_2 & \Gamma_{23} \\ 0 & 0 & k_3 \end{bmatrix}, \qquad (17)$$

where the k_1, k_2 and k_3 represent the extensions–contractions along the x_1, x_2 and x_3 co-ordinate axes (includes volume change if $k_1 \cdot k_2 \cdot k_3 \neq 1$), and the off-diagonal terms (Γ_{ij}) represent elements of shear deformation. If we consider a left-handed co-ordinate system, Γ_{12} reflects a wrench in the x_1 direction, and Γ_{13} and Γ_{23} correspond to a thrust in the x_1 and x_2 directions, respectively (Fig. 5).

The deformation components involved can be illustrated by their individual deformation matrices:

$$\begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}, \quad \begin{bmatrix} 1 & \gamma_W & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

pure shear ± volume change; wrench in the x_1 direction

$$\begin{bmatrix} 1 & 0 & \gamma_{T1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \gamma_{T2} \\ 0 & 0 & 1 \end{bmatrix}. \qquad (18)$$

thrust in the x_1 direction; thrust in the x_2 direction

It can be shown (Tikoff & Fossen in press) that the simultaneous combination of these deformations gives the deformation matrix (17) with

$$\Gamma_{12} = \frac{\gamma_W(k_1 - k_2)}{\ln(k_1/k_2)} \qquad (19a)$$

$$\Gamma_{13} = \frac{\gamma_{T1}(k_1 - k_3)}{\ln(k_1/k_3)} + \frac{\gamma_{T2}\gamma_W(k_1 - k_2)}{\ln(k_1/k_2)\ln(k_2/k_3)} + \frac{\gamma_{T2}\gamma_W(k_3 - k_1)}{\ln(k_2/k_3)\ln(k_1/k_3)} \qquad (19b)$$

$$\Gamma_{23} = \frac{\gamma_{T2}(k_2 - k_3)}{\ln(k_2/k_3)}. \qquad (19c)$$

With (19a)–(19c), matrix (17) contains all the information about the deformation, such as the orientations, magnitudes and rotations of the principal directions in space.

The calculation of strain paths in three dimensions is similar to their calculation in plane strain, as shown above. The deformation matrix (17) above is not restricted to combinations of thrusting in the x_1 and x_2 directions and wrenching in the x_1 direction, in addition to the general coaxial strain and/or volume change. Switching the orientations of x_1 and x_2 gives a combination of thrusting in the x_1 and x_2 directions and wrenching in the x_2 direction. Switching x_1 and x_3 (making x_1 vertical) gives a combination of two vertical shears and wrenching in the x_2 direction. Making x_2 the vertical direction gives wrenching and thrusting in the x_1 direction and vertical shear in the x_2–x_3 plane. Hence, the exact deformation matrix for a large variety of complex deformations can be found using (17) and (19a)–(19c). The restrictions are that the deformation is homogeneous, that the coaxial principal deformation axes are perpendicular to the shear planes, and that the shear planes are mutually orthogonal.

VORTICITY

The kinematic vorticity number, W_k (Truesdell 1953), is a useful measure of non-coaxiality, particularly in cases where it can be extracted from naturally deformed rocks (e.g. Passchier 1987, 1990, Passchier & Urai 1988, Vissers 1989), and is discussed by Means *et al.* (1980). Its relationship with simultaneous coaxial deformation and simple shearing is

$$W_k = \gamma\{2(\ln k_1)^2 + 2(\ln k_2)^2 + \gamma^2\}^{-1/2} \qquad (20)$$

for two dimensions (10), and

$$W_k = \{(\gamma_{T1})^2 + (\gamma_W)^2 + (\gamma_{T2})^2\}^{1/2} \cdot \{2(\ln k_1)^2 + 2(\ln k_2)^2 + 2(\ln k_3)^2 + (\gamma_{T1})^2 + (\gamma_{T2})^2 + (\gamma_W)^2\}^{-1/2} \qquad (21)$$

for three dimensions (17) (for derivation, see Tikoff & Fossen in press). Combinations of coaxial deformation and simple shearing(s) produce vorticity numbers between 0 and 1, where $W_k = 0$ for pure shearing and $W_k = 1$ for simple shearing.

STRAIN PATHS

There are an infinite number of strain paths that can produce a particular state of finite strain. The simplest is *steady flow*, which has been assumed in this paper. By steady flow, we mean that the incremental strain matrix does not change during the deformation history, i.e. the principal instantaneous stretching directions and the kinematic vorticity number (W_k) stay constant. The *strain path* can then be studied by choosing a fixed strain increment and calculating the deformation matrix after each increment. For n increments and steady flow deformation, the incremental deformation matrix for a simultaneous pure shearing, simple shearing and dilating can be written in terms of the total pure shear and simple shear components as

$$\mathbf{D}_{\text{incr}} = \begin{bmatrix} k_{1_{\text{incr}}} & \Gamma_{\text{incr}} \\ 0 & k_{2_{\text{incr}}} \end{bmatrix} = \begin{bmatrix} (k_{1_{\text{tot}}})^{1/n} & n^{-1}\gamma_{\text{tot}} \dfrac{(k_{1_{\text{tot}}})^{1/n} - (k_{2_{\text{tot}}})^{1/n}}{\ln (k_{1_{\text{tot}}}/k_{2_{\text{tot}}})^{1/n}} \\ 0 & (k_{2_{\text{tot}}})^{1/n} \end{bmatrix}. \quad (22)$$

Matrix (22) gives the exact incremental strain for any steady flow formed by a simultaneous combination of pure shearing and simple shearing. A similar matrix can be found for three-dimensional strain by using the relationships $\gamma_{\text{incr}} = (\gamma_{\text{total}})/n$, and $k_{\text{incr}} = (k_{\text{total}})^{(1/n)}$. The relationship between the incremental and finite deformation matrix for stready flow is

$$\mathbf{D}_{\text{total}} = (\mathbf{D}_{\text{incr}})^n. \quad (23)$$

The orientations, magnitudes and rotations of the principal strains can be calculated precisely at any step (see the Appendix), and thus mapped throughout the deformation.

Non-steady flow deformation paths can be modeled by changing the incremental deformation matrix gradually. For example, if the progressive volume change demonstrably had a decreasing role during a shearing event, the incremental matrix can be changed gradually throughout the matrix pre-multiplication process. However, it is usually hard to obtain information about the flow history of a particular deformation, and steady flow may be considered a reasonable 'standard of reference' for geological modelling (cf. Passchier 1990).

APPLICATION TO TRANSPRESSION–TRANSTENSION

Transpression and transtension are two closely related types of deformation which, by definition, involve the simultaneous combination of simple shearing and pure shearing. If we adopt the constraints used by Sanderson & Marchini (1984) (free upper horizontal surface, fixed lower horizontal surface, no volume change), transpression–transtension can mathematically be modelled as a combination of a vertical simple shear (wrench) in the x_1 direction and a pure shear in the x_2–x_3 plane (Fig. 6). The compatibility problem normally involved with such deformation is reduced by letting the free upper surface represent the surface of the Earth.

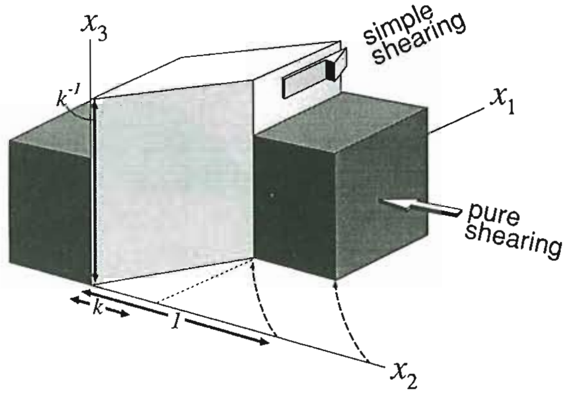

Fig. 6. Schematic illustration of transpression as a combination of simultaneous simple shear and pure shear. Based on diagram by Sanderson & Marchini (1984).

Sanderson & Marchini (1984), in their discussion on *finite* deformations in transpression zones, factorized the total deformation matrix into a pure shear and a simple shear component:

$$\mathbf{D} = \begin{bmatrix} 1 & \gamma_{\text{p,s}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & k & 0 \\ 1 & 0 & k^{-1} \end{bmatrix} = \begin{bmatrix} 1 & k\gamma_{\text{p,s}} & 0 \\ 0 & k & 0 \\ 0 & 0 & k^{-1} \end{bmatrix}. \quad (24)$$

Whereas such a factorization is a good choice for finite deformations, the *progressive* deformation involved in transpression–transtension zones is easier and more accurately modeled by using a simplified version of matrix (17):

$$\mathbf{D} = \begin{bmatrix} 1 & \Gamma & 0 \\ 0 & k & 0 \\ 0 & 0 & k^{-1} \end{bmatrix} = \begin{bmatrix} 1 & \dfrac{\gamma(1-k)}{\ln (k^{-1})} & 0 \\ 0 & k & 0 \\ 0 & 0 & k^{-1} \end{bmatrix}, \quad (25)$$

where k is the horizontal pure shear component in the x_2 direction (perpendicular to the shear plane) and γ is the shear strain in the x_1 direction (horizontal). $k > 1$ and $\gamma \neq 0$ gives transtension, whereas $k < 1$ and $\gamma \neq 0$ gives transpression.

Strain geometry

Any simultaneous combination of the simple shearing and pure shearing in the system indicated in Fig. 6 gives rise to a unique state of strain. The *shape* of the strain ellipsoid can be illustrated on a contoured, logarithmic Flinn diagram (Fig. 7), and it is clear that transtensional deformation gives rise to constrictional strain, whereas transpressional deformation results in flattening strain. This important fact, which was also pointed out by Sanderson & Marchini (1984), predicts $S(L)$-dominated fabrics in transpressional shear zones, and $L(S)$-fabrics where the deformation is transtensional. Note, how-

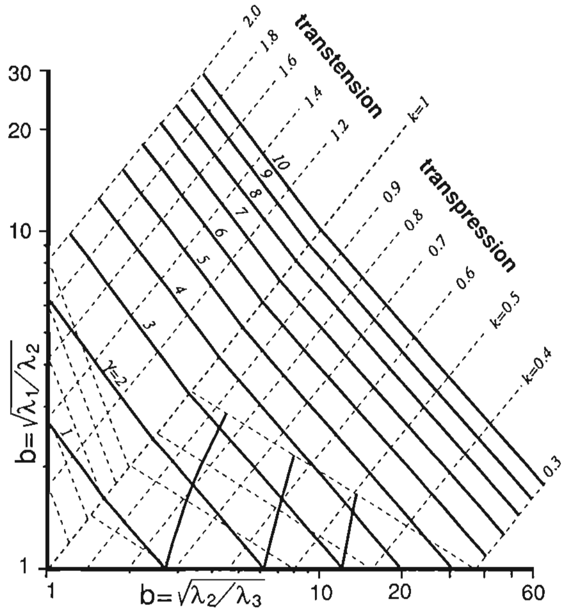

Fig. 7. Logarithmic Flinn diagram contoured for γ (simple shear) and k (pure shear) values, using deformation matrix (equation 25) (transpression–transtension). Pure shear (k) contours are similar to those shown by Sanderson & Marchini (1984, fig. 2), but simple shear contours (γ) are different because of different definitions of γ (see discussion in text).

ever, that a similar Flinn diagram presented by Sanderson & Marchini is different from Fig. 7 because the shear strain used by Sanderson & Marchini (here called $\gamma_{p,s}$) is different from our γ.

The three-dimensional *orientation* of the strain ellipse can be mapped for various combinations of pure shear and simple shear components (Fig. 8). λ_2 is vertical for wrenching deformations dominated by simple shearing. However, for highly transpressional deformation, the λ_1 axis is vertical, whereas λ_3 is the vertical principal strain axis for highly transtensional deformation. For steady flow, and $W_k \leqslant 0.81$, either λ_1 (for transpression) or λ_3 (for transtension) is the vertical principal axis throughout deformation (Fig. 8). However, for $1 > W_k > 0.81$,

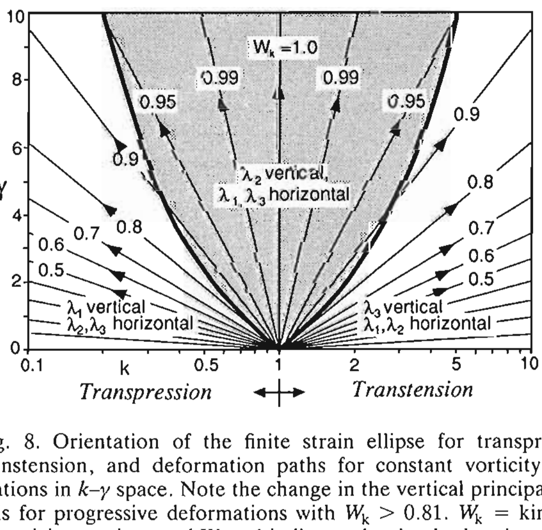

Fig. 8. Orientation of the finite strain ellipse for transpression–transtension, and deformation paths for constant vorticity deformations in k–γ space. Note the change in the vertical principal strain axis for progressive deformations with $W_k > 0.81$. W_k = kinematic vorticity number, and $W_k = 1$ indicates the simple shearing path.

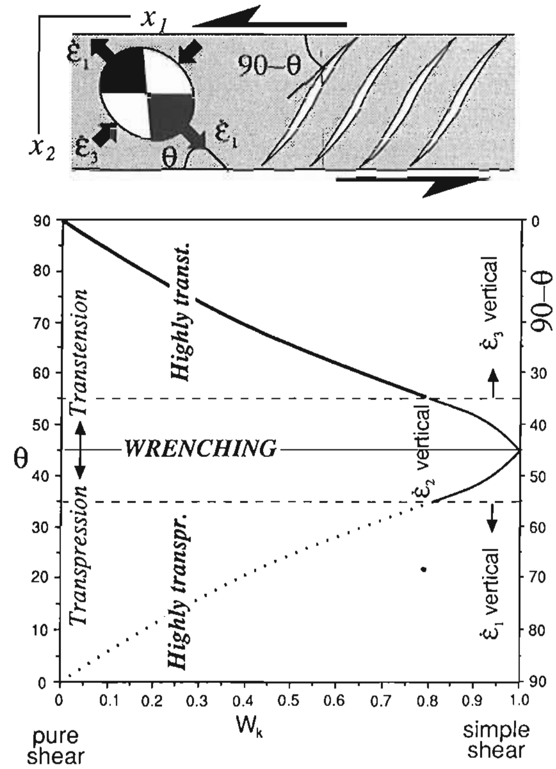

Fig. 9. Orientation of the maximum *instantaneous* stretching direction ($\dot{\varepsilon}_1$) with varying kinematic vorticity number, for transpression and transtension. At $W_k = 0.81$, the maximum instantaneous stretching direction becomes vertical for transpression, and the orientation of $\dot{\varepsilon}_2$ is shown as a dashed line where $\dot{\varepsilon}_1$ is vertical.

λ_2 starts out as the vertical principal strain axis, but switches position with λ_1 (for transpression) or λ_3 (for transtension) during deformation according to the constant vorticity paths in Fig. 8.

The two principal strain axes in the horizontal plane are oblique to the x_1 and x_2 axes, depending on the relative amounts of pure shear and simple shear. The angle θ' that the longest of the horizontal principal strain axes makes with the x_1 (shear) direction is given by the equation

$$\theta' = \arctan\left(\frac{\lambda_{\text{hormax}} - \Gamma^2 - 1}{k\Gamma}\right). \qquad (26)$$

A particularly useful piece of information is provided by the direction of the instantaneous stretching axes, and in particular the angle (α) between the maximum instantaneous stretching direction ($\dot{\varepsilon}_1$) and the shear zone (Fig. 9). If tension gashes are present in the transpression–transtension zone, they may be used to estimate the vorticity number W_k of the deformation (the relative amount of pure shearing and simple shearing), using the graph shown in Fig. 9. The same graph is in some cases also applicable to larger scale features, such as extensional faults, dike swarm systems, thrust faults and fold axes formed in transpression–transtension zones. However, once formed, all such features will rotate during further deformation, as discussed below.

Changes in angles and lengths

The formulas presented by Sanderson & Marchini (1984) must be modified when using matrix (25) for either finite or progressive strain considerations by substituting for their shear strain (here called $\gamma_{p,s}$) with the expression:

$$\gamma_{p,s} = \frac{\gamma(1-k)}{k \ln k^{-1}} = \frac{\Gamma}{k}. \qquad (27)$$

For example, the orientation of the vector or the passive line after deformation then is defined by the equation

$$\cot \phi' = k^{-1} \cot \phi + \Gamma k^{-1}$$
$$= k^{-1} \cot \phi + \frac{\gamma(1-k)}{k \ln (k^{-1})}, \qquad (28)$$

where ϕ is the initial angle, and ϕ' is the new angle between the line and the x_1 axis. Similarly, the extension (quadratic elongation) of a horizontal line is given by the equation

$$\lambda_{hor} = (\cos \phi + \Gamma \sin \phi)^2 + k^2 \sin^2 \phi. \qquad (29)$$

Any line in space (not necessarily horizontal) with initial direction given by the unit vector $l = (a, b, c)$ transforms into the new vector l' where

$$l' = \begin{bmatrix} a' \\ b' \\ c' \end{bmatrix} = \begin{bmatrix} a + b\Gamma \\ kb \\ c/k \end{bmatrix} \qquad (30)$$

and the actual quadratic elongation of the line is

$$\lambda = \frac{c^2}{k^2} + b^2k^2 + (a + b\Gamma)^2. \qquad (31)$$

The angle α between the line and the x_1 axis (shear direction) is given by the equation

$$\alpha = \arccos \frac{a'}{\sqrt{\lambda}}. \qquad (32)$$

These equations can be used to study the reorientation of fold axes and other linear structures that behave as passive markers, or nearly so, during transpression–transtension. Fold axes that initiate at an angle to the shear plane (x_1–x_3 plane) will rotate during deformation. For simple shearing alone, fold axes are well known to rotate towards x_1 with increasing shear strain. Using (25), we can now study the behavior of folds in transtensional–transpressional regimes.

Four initial orientations have been chosen as examples: 090/00, 135/00, 090/45 and 135/45. The kinematic vorticity number, w_k, has been kept fixed at 0.75 throughout the deformation. As a measure of the total three-dimensional strain, we have used the unit

$$\bar{\varepsilon}_s = \frac{\sqrt{3}}{2} \bar{\gamma}_0,$$

where $\bar{\gamma}_0$ is the natural octahedral unit shear (cf. Hossack 1968).

The results (Figs. 10 and 11) indicate that horizontal material lines will remain horizontal during both transpressional and transtensional deformation. A horizontal line with initial orientation 90° to the shearing (x_1) direction rotates very similarly under simple shearing and transpressional deformation. Other horizontal lines, however, rotate more slowly towards the x_1 axis with transpression than with simple shearing. Transtension makes all horizontal lines rotate slower than during simple shearing, and the lines rotate towards a horizontal, asymptotic line which makes about 24° with the x_1 axis when $W_k = 0.75$. This angle is defined by the apophysis (cf. Bobyarchick 1986) of the flow which is not parallel to the x_1 or x_3 axes. This apophysis, which is horizontal, is the eigenvector corresponding to the largest eigenvalue of the velocity gradient field (e.g. Bobyarchick 1986, Tikoff & Fossen in press). The angle between this oblique flow apophysis and x_1 can be shown to be

$$\arctan \left(\frac{\ln k}{\gamma} \right). \qquad (33)$$

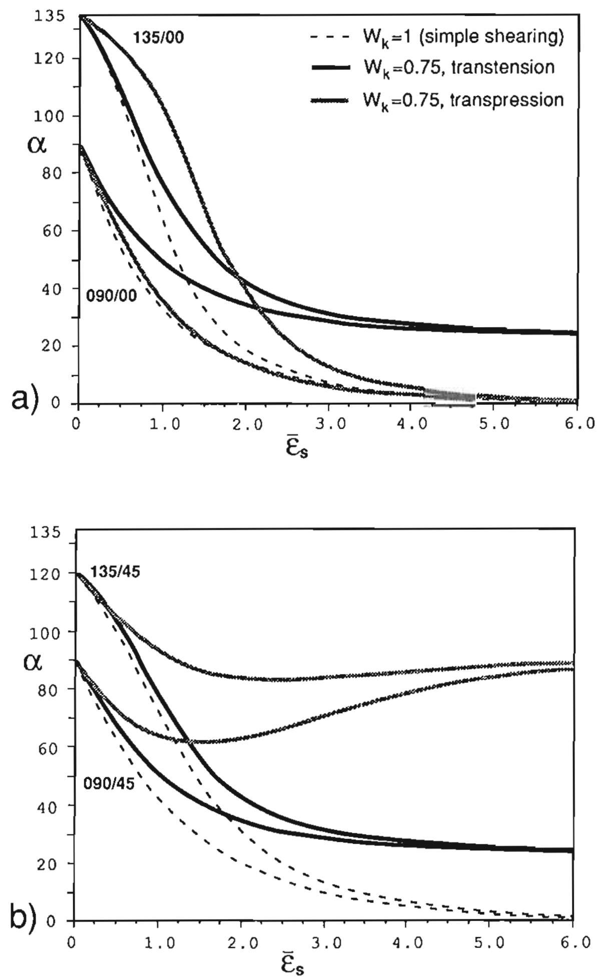

Fig. 10. Change in orientation of passively deforming lines (e.g. fold axes) for simple shear wrenching (dashed line) and transtension–transpression. In (a) the initial lines are 135/00 and 090/00, and in (b) 135/45 and 090/45. α = angle between the line and the shear direction (x_1). See text for discussion.

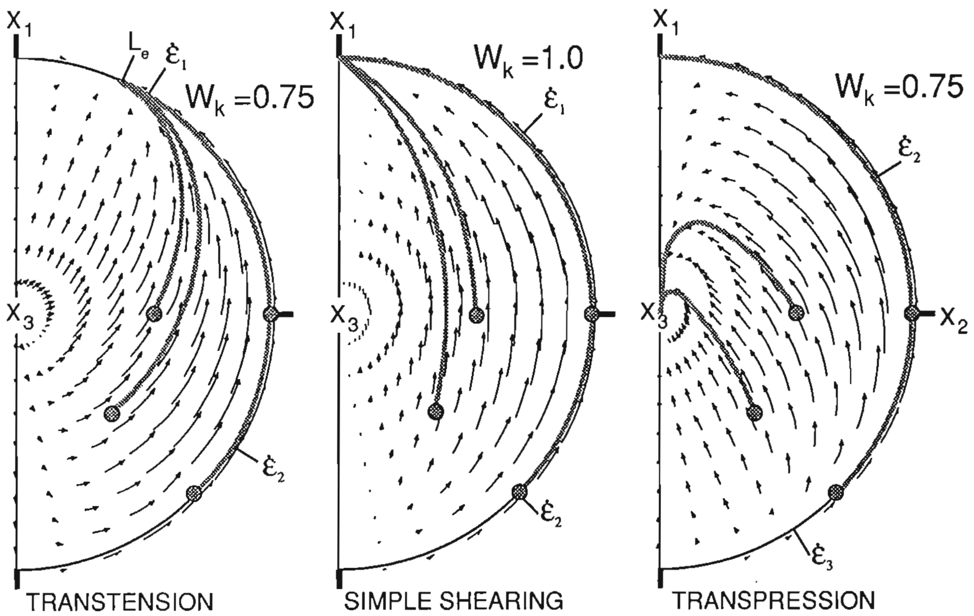

Fig. 11. Stereographic illustration of the progressive rotation of passive line markers for transpression–transtension (W_k = 0.75) and simple shearing (wrenching). Arrows indicate the result of simple shear strains of 0.25 combined simultaneously with $k = 1.1165456$ (transpression), $k^{-1} = 1.1165456$ (transtension), and $k = 1$ (simple shear) to perform the rotations. The lengths of the arrows therefore indicate the rates of rotation in the different fields of the stereograms. The total deformation paths of the four lines discussed in Figs. 10(a) & (b) are shown. $\dot{\varepsilon}_1$ is the largest instantaneous stretching direction, and L_e is the flow apophysis in the direction defined by the eigenvector corresponding to the largest eigenvalue of the velocity gradient tensor L. L_e is parallel to x_1 and x_2 for simple shearing and transpression, respectively.

This angle, which varies from 90° for pure shearing to 0° for simple shearing, is also the value which the maximum horizontal principal strain axis approaches (but never reaches) with increasing strain. This implies that for transtension, for which strong linear fabrics are expected, there is always a considerable angle between the stretching lineation and the shear direction (>24° for $W_k = 0.75$).

All lines inclined away from the x_1–x_2 shear plane rotate more slowly during transtension–transpression than during simple shearing (Fig. 10b). For transtension, the angle approaches the asymptotic value given by (33), whereas for transpression the inclined lines start to rotate away from x_1 toward the vertical x_3 axis at some point. This change in rotation direction occurs only in transpression when λ_1 is vertical (Fig. 8), and is due to the fact that λ_1 and λ_2 are relatively close in magnitude. It can be seen from the flow pattern in Fig. 11(c) that this effect is largest for initially shallowly plunging lines in the first (and third) quadrant(s) of the stereogram. These considerations demonstrate that fold hinges and other linear features that behave in a passive manner can never become parallel to the shear direction in transtensional shear zones, and only if they originated as perfectly horizontal lines in transpressional shear zones.

CONCLUSIONS

Progressive, as well as finite deformation, can be modelled using a single matrix, given in terms of the pure shearing (k) and simple shearing (λ) components. The strain path can be calculated using the relationships $\gamma_{incr} = (\gamma_{total})/n$ and $k_{incr} = (k_{total})^{(1/n)}$, where γ and k are, respectively, simple shear and pure shear factors. These solutions give identical finite deformations and deformation paths to those described by Ramberg's strain rate- and time-dependent equations, but are simpler to use.

Application of this strain theory shows that transpressional deformation produces flattening or planar fabrics, whereas transtensional deformation results in strongly linear and constrictional fabrics. Assuming steady flow, any finite state of strain is the result of a unique combination of simultaneous pure shearing and simple shearing. The orientations of the principal axes of the finite strain ellipse depend on the vorticity as well as on whether the deformation is highly transpressive, transtensive, or simple shear-dominated wrenching. Furthermore, the characteristic patterns of rotation of material lines (e.g. fold axes and lineations) are shown to be significantly different for transtension and transpression. In the former case the lines rotate towards the flow asymptote that is oblique to the x_1 axis, and with transpression, non-horizontal lines eventually rotate towards the vertical (x_3) axis (vertical flow asymptote) and away from the shear direction. Also, strain histories different from steady flow conditions can be modeled using gradually changing incremental deformation matrices.

Acknowledgements—This work was supported by the Norwegian Research Council for Science and the Humanities (NAVF grant No. 440.89/061) for H. Fossen, and a Stanwood Johnson fellowship (University of Minnesota) for B. Tikoff. Reviews by W. Means and D. Sanderson helped improve and clarify the manuscript. P. Hudleston and C. Teyssier are thanked for helpful conversations.

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APPENDIX

The deformation matrix for plane strain (10) and three dimensions (17 and 19a–19c) has been tested numerically against a computer program which gives an approximate solution by successively premultiplying small increments of pure shear and simple shear(s). The solution converged towards the deformation matrix as the size of the increments was decreased. The mathematical derivations of these matrices are shown in Tikoff & Fossen (in revision).

The volume change (Δ) involved in the deformation is simply $\det(\mathbf{D}) - 1 \times 100\%$, and is negative for volume decrease and positive for volume increase. To find the geometry of the strain ellipsoid from the two- or three-dimensional unified deformation matrix, form the matrix $\mathbf{DD}^{\mathrm{T}}$. The eigenvalues of this matrix (always 2 for 2×2 matrices, and 3 for 3×3 matrices for geologically realistic deformations) are the quadratic principal strain magnitudes, e.g. $\lambda_1 = (1 + e_1)^2$, and their corresponding eigenvectors give the directions of the principal axes in the deformed state. For two dimensions the eigenvalues (lengths of the strain ellipse axes) are given by the formula

$$\lambda = \frac{\Gamma^2 + k_1^2 + k_2^2 \pm \sqrt{-4k_1^2k_2^2 + (\Gamma^2 + k_1^2 + k_2^2)^2}}{2} \qquad \text{(A1)}$$

and the corresponding eigenvectors can be expressed as

$$\mathbf{e} = \begin{bmatrix} \dfrac{-k_2\Gamma}{\Gamma^2 + k_1^2 - \lambda} \\ 1 \end{bmatrix}. \qquad \text{(A2)}$$

For three dimensions the eigenvalues and eigenvectors are more easily solved for numerically. The angle θ' between the largest principal strain axis and the shear (x_1) direction is

$$\theta' = \arccos(e_{11}) \qquad \text{(A3)}$$

where e_{11} is the first component of the normalized eigenvector of $\mathbf{DD}^{\mathrm{T}}$ corresponding to λ_1 (the normalized form of a vector $\mathbf{v}$ is $\mathbf{v}/(\mathbf{v}^{\mathrm{T}}\mathbf{v})^{1/2}$). In both two and three dimensions, one can study the change in orientation of any line from its initial orientation, given by the unit vector $\boldsymbol{l}$ (made up of the direction cosines of the line) to the new direction $\boldsymbol{l}'$ by the transformation

$$\boldsymbol{l}' = \mathbf{D}\boldsymbol{l}. \qquad \text{(A4)}$$

The angle of rotation (φ) of this line can be found from the formula

$$\cos\phi = \frac{\boldsymbol{l}^{\mathrm{T}}\boldsymbol{l}'}{\sqrt{\boldsymbol{l}'^{\mathrm{T}}\boldsymbol{l}'}} \qquad \text{(A5)}$$

and the quadratic extension (λ) of the line is simply

$$\lambda = \boldsymbol{l}'^{\mathrm{T}}\boldsymbol{l}'. \qquad \text{(A6)}$$

The angle β between the largest principal strain axis ($\mathbf{e}_1$) and the line is

$$\beta = \arccos(\mathbf{e}_1^{\mathrm{T}}\boldsymbol{l}'). \qquad \text{(A7)}$$

where $\mathbf{e}_1$ is the normalized eigenvector corresponding to the largest eigenvalue (λ_1) of $\mathbf{DD}^{\mathrm{T}}$. The new vector $\boldsymbol{l}'$ has, in general, a length different from unity, but may be normalized to reveal the new direction cosines with respect to the x_1, x_2 and x_3 co-ordinate axes, respectively.

Similarly, if $\mathbf{p}$ is the pole to a plane prior to deformation, the new orientation of the plane is given by $\mathbf{p}'$, where

$$\mathbf{p}' = \mathbf{p}\mathbf{D}^{-1}. \qquad \text{(A8)}$$

The rotation of $\mathbf{p}$ equals the rotation of the plane, and can be found by using equation (A5).